Z. Sylvan – 17/2/11 – Special Lagrangians and SYZ

SYZ: mirror symmetry = duality between special Lagrangian torus fibrations on CYs.

Def. An (almost) CY mfld = Kähler mfld + nonvanishing holomorphic $(n,0)$-form Ω.
Strict CY = $\Omega \wedge \bar\Omega$ = const. $dvol_g$

Def. A Lagrangian submfd L is special if $\operatorname{Im} \Omega_{|L} = 0$.

Hence $\Omega_{|L} = \psi \, dvol_{g|L}$ (strict case: ψ = constant; almost CY: $\psi \in C^\infty(L, \mathbb{R}_+)$).

We looked at $\arg(\Omega_{|L})$ as a way of grading Floer theory;
for L SLag, $\arg(\Omega_{|L})$ = constant, so lift always exists (ie. $\mu_L = 0$).
[among other facts, SLags in strict CYs are minimal submflds ...].

Deformations of special Lagrangians

Thm (McLean) Let X Calabi-Yau, $L \subset X$ special Lag. (compact)
$\Rightarrow$ special Lag. deformations of L in X form a smooth manifold B of dimension $b_1(L)$, with $T_L B \cong \mathcal{H}^1(L)$ (harmonic 1-forms)

Note: if L is a torus w/ suff'tly nice metric, then harmonic 1-forms have no zeroes, so locally n-dim! family gives a SLag fibration.

Pf: 1) look at infinitesimal deformations: assume $\varphi_t : L \to X$, $t \in (-\varepsilon, \varepsilon)$, φ_0 = inclusion,

$\left.\frac{d\varphi_t}{dt}\right|_{t=0} = v$ normal vector field along L, $v \in C^\infty(NL)$; want $\varphi_t(L)$ SLag.

- Symplectic geometry: know $\iota_v \omega$ must be a closed 1-form on L

namely $0 = \frac{d}{dt}(\varphi_t^* \omega)_{|t=0} = L_v \omega = d \underbrace{\iota_v \omega}_{\iota_v \omega \,=\, \text{1-form on } L}$

using
$$NL \xrightarrow{\sim} T^*L$$
$$v \pmod{TL} \longmapsto \iota_v \omega_{|TL}$$

- Similarly, to preserve $\varphi_t^*(\mathrm{Im}\,\Omega) = 0$:

$$0 = \frac{d}{dt}\left(\varphi_t^* \mathrm{Im}\,\Omega\right)_{|t=0} = \mathcal{L}_v \mathrm{Im}\,\Omega = d(\iota_v \mathrm{Im}\,\Omega)$$
(since Ω closed).

<u>Conclusion:</u> v gives SL deformation $\iff$ $\left.\begin{array}{l}\iota_v\omega \text{ 1-form on } L \\ \iota_v \mathrm{Im}\,\Omega \ (n-1)\text{-form on } L\end{array}\right\}$ both <u>closed</u>.

Recall $\omega : NL \xrightarrow{\sim} T^*L$, let $\beta = -\iota_v\omega$; let $\tilde\beta = \iota_v \mathrm{Im}\,\Omega$.

<u>Pointwise:</u> let $\partial_{x_1} \dots \partial_{x_n}$ orthonormal basis for T_pL
extend by $\partial_{y_j} = J\partial_{x_j}$ to o.n. basis for T_pX

Then $\omega = \sum dx_i \wedge dy_i$, $g = \sum dx_i^2 + dy_i^2$, $\Omega = c \prod dz_i$

Now $v = \sum a_i \partial_{y_i}$ $\Rightarrow$ $\beta = \sum a_i\, dx_i$
$$\tilde\beta = \sum (-1)^{i-1} a_i\, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n = *\beta$$
(since $\Omega = dx_1 \wedge \dots \wedge dx_n + i\sum dx_1 \wedge \dots \wedge dy_j \wedge \dots \wedge dx_n + \dots$)

So $d\beta = d\tilde\beta = 0 \iff d\beta = d^*\beta = 0 \iff \beta = -\iota_v\omega \in \mathcal{H}^1(L)$.

2) Show deform$^{\text{n}}$ space $\mathcal{B}$ is a smooth manifold:
(<u>Issue</u>: we only studied the <u>linearization</u> of the PDE expressing condition that L_t be slag).

Nearby deform$^{\text{s}}$ of L $\longleftrightarrow$ small $v \in \Gamma(NL)$
$\exp(v)(L)$ $\longleftrightarrow$ v

If v = normal vector field, then
$$\exp(v)^*\omega = \int_0^1 \exp(tv)^* \mathcal{L}_{v_t}\omega = d\left(\int_0^1 \exp(tv)^* \iota_{v_t}\omega\right)$$
is always exact

Similarly, $\exp(v)^*(\mathrm{Im}\,\Omega)$ is always exact.

So: $W^k(L, NL) \longrightarrow W^{k-1}(L, T^*L)_{exact} \oplus W^{k-1}(L, \Lambda^{n-1}T^*L)_{exact}$
(k large enough) $\quad v \longmapsto s(v) = \left(-\exp(v)^*\omega,\ \exp(v)^*\mathrm{Im}\,\Omega\right)$

s nonlinear map between Banach spaces; $\quad B \cong s^{-1}(0)$.

So: want to show $ds(0)$ is onto.

Point: by step 1, $\quad ds(0)(v) = (-d\iota_v\omega,\ d\iota_v \operatorname{Im}\Omega) = (d\beta,\ d{*}\beta)$

This is onto exact forms by Hodge decomposition. ◀

☆ So: if L_0 a sufficiently nice SL torus, we might ideally get a SL fibration $X \to B$ by tori.

B carries two natural affine structures, from

$$T_L B \xrightarrow{\sim} H^1(L,\mathbb{R}) \quad \text{and} \quad T_L B \xrightarrow{\sim} H^{n-1}(L,\mathbb{R})$$
$$v \longmapsto [\iota_v\omega] \qquad\qquad v \longmapsto [\iota_v \operatorname{Im}\Omega].$$

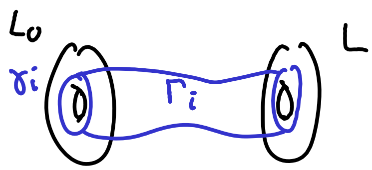
$$(\supseteq \text{integer lattices } H^1(L,\mathbb{Z}),\ H^{n-1}(L,\mathbb{Z}))$$

Mirror symmetry will be an interchange in these affine structures.

- Affine coords. on B:

  let $\gamma_1,\dots,\gamma_n$ basis for $H_1(L)\simeq\mathbb{Z}^n$

  $\gamma_1^\alpha,\dots,\gamma_n^\alpha$ basis for $H_{n-1}(L)$



Γ_i = cylinder traced by γ_i (2-dim^l)

Γ_i^α $\qquad\qquad\qquad\qquad\ \ \gamma_i^\alpha$ (n-dim^l)

$\Rightarrow$ affine coords. $\quad x_i = \int_{\Gamma_i}\omega\ ; \quad \left[dx_i(v) = \int_{\gamma_i}\iota_v\omega\right]$

$$x_i^\alpha = \int_{\Gamma_i^\alpha}\operatorname{Im}\Omega \quad \left[dx_i^\alpha(v) = \int_{\gamma_i^\alpha}\iota_v\operatorname{Im}\Omega\right].$$

This is a local construction;

globally, holonomy in $\mathbb{R}^n \rtimes GL(n,\mathbb{Z})$.